Vertex-magic Total Labelings of Union of Generalized Petersen Graphs and Union of Special Circulant Graphs

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Abstract. Let $G$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$, and let $n = |V(G)|$ and $e = |E(G)|$. A vertex-magic total labeling (VMTL) of a graph is defined as a one-to-one mapping taking the vertices and edges onto the set of integers $\{1, 2, \ldots, n + e\}$, with the property that the sum of the label on a vertex and the labels on its incident edges is a constant independent of the choice of vertex. In this paper, we present the vertex magic total labeling of disjoint union of $t$ generalized Petersen graphs $\bigcup_{j=1}^{t} P(n_j, m_j)$, and disjoint union of $t$ special circulant graphs $\bigcup_{j=1}^{t} C(n, 1, m_j)$.

1 Introduction

In this paper all graphs are finite, simple, and undirected. The graph $G$ has vertex set $V = V(G)$ and edge set $E = E(G)$, and let $n = |V(G)|$ and $e = |E(G)|$.

MacDougall et al. [4] introduced the notion of a vertex-magic total labeling. The vertex-magic total labeling of a graph $G$ is a one-to-one mapping from $V \cup E$ onto the integers $1, 2, \ldots, n + e$ such that for every vertex $x \in V$ there is a constant $k$ so that for every vertex $x$,

$$w_\lambda(x) = \lambda(x) + \sum_{y \in N(x)} \lambda(xy) = k,$$

where $N(x)$ is the set of all vertices $y$ that adjacent to $x$. The constant $k$ is called the magic constant for $\lambda$ and $w_\lambda(x)$ is called the weight of $x$ under labeling $\lambda$.

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The generalized Petersen graph \( P(n, m), n \geq 3, 1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor \), is a regular graph with an outer \( n \)-cycle \( u_1, u_2, \ldots, u_n \), a set of \( n \) spokes \( u_i v_i, 1 \leq i \leq n \), and \( n \) inner edges \( v_i v_{i+m} \) with indices taken modulo \( n \). The standard Petersen graph is \( P(5,2) \). Generalized Petersen graphs were first defined by Watkins [7].

Let \( 1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq \lfloor n/2 \rfloor \), where \( n \) and \( a_i (i = 1, \ldots, k) \) are positive integers. A circulant graph \( C_n(a_1, a_2, \ldots, a_k) \) is a regular graph with \( V = \{v_0, v_1, \ldots, v_{n-1}\} \) and \( E = \{(v_i v_{i+a_j}) \mod (n-1) | i = 0, 1, \ldots, n-1, j = 1, 2, \ldots, k\} \).

If a regular graph \( G \) possesses a vertex-magic total labeling, we can create a new labeling \( \lambda' \) from \( \lambda \) by setting
\[
\lambda'(x) = n + e + 1 - \lambda(x)
\]
for every vertex \( x \), and
\[
\lambda'(xy) = n + e + 1 - \lambda(xy)
\]
for every edge \( xy \). Clearly, \( \lambda' \) is also a one-to-one mapping from the set \( V \cup E \) to \( 1, 2, \ldots, n + e \) and we call \( \lambda' \) as the dual of \( \lambda \). If \( r \) is the degree of each vertex of the regular graph \( G \), then
\[
k' = (r + 1)(n + e + 1) - k
\]
is the new magic constant for \( \lambda' \).

Since the introduction of labeling, there have been several results on vertex magic total labeling of particular classes of graphs. For example, MacDougall et al.[4] proved that the cycle \( C_n \) for \( n = 3 \), path \( P_n \) for \( n = 2 \), complete graph \( K_n \) for odd \( n \), and complete bipartite graph \( K_{n,n} \) for \( n > 1 \), have vertex-magic total labeling. Baca, Miller, and Slamin [1] proved that for \( n = 3, 1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor \), every generalized Petersen graph \( P(n, m) \) has vertex-magic total labelings with the magic constant \( k = 9n + 2, k = 10n + 2, \) and \( k = 11n + 2 \). Balbuena et al [2] proved that for odd \( n \geq 5 \) and \( m \in 2,3,\ldots,(n-1)/2 \), circulant graphs \( C_n(1, m) \) has a super vertex-magic total labeling with \( k = (17n + 5)/2 \). The complete survey of the known results on vertex-magic total labeling of graphs can be found in [3].

Most of the known results are concerning on vertex-magic total labeling of connected graphs. For the case of disconnected graph, Wallis [6] proved Theorem 1. Slamin et al. [5] proved that the 2 copies of generalized Petersen graphs \( 2P(n, m) \) has a vertex-magic total labeling with the magic constant \( k = 19n + 2 \) and \( k = 21n + 2 \).

**Theorem 1** [6] Suppose \( G \) is a regular graph of degree \( r \) which has a vertex-magic total labeling. (i) If \( r \) is even, then \( tG \) is vertex-magic whenever \( t \) is an odd positive integer. (ii) If \( r \) is odd, then \( tG \) is vertex-magic for every positive integer \( t \).
2 Main Result

As mentioned in the introduction, Slamin et al. [5] had given the construction of vertex-magic total labeling for the 2 copies of generalized Petersen graphs $2P(n, m)$. We found that this label can be extended to $t$-copies of generalized Petersen graphs $tP(n, m)$.

In this section, we present a construction of a vertex-magic total labeling for a more general case. We construct the vertex-magic total labeling for the union of $t$ generalized Petersen graphs, $\bigcup_{j=1}^{t} P(n_j, m_j)$. The union of $t$ generalized Petersen graphs $\bigcup_{j=1}^{t} P(n_j, m_j)$ has a vertex set $V(\bigcup_{j=1}^{t} P(n_j, m_j)) = \{u_i^j \mid 1 \leq i \leq n_j, 1 \leq j \leq t\} \cup \{v_i^j \mid 1 \leq i \leq n_j, 1 \leq j \leq t\}$ and the edge set $E(\bigcup_{j=1}^{t} P(n_j, m_j)) = \{u_i^j u_{i+1}^j \mid 1 \leq i \leq n_j, 1 \leq j \leq t\} \cup \{u_i^j v_i^j \mid 1 \leq i \leq n_j, 1 \leq j \leq t\} \cup \{v_i^j v_{i+m_j}^j \mid 1 \leq i \leq n_j, 1 \leq j \leq t\}$. In the next Theorem we prove that union of $t$ generalized Petersen graphs has a vertex-magic total labeling.

**Theorem 2** For $n_j \geq 3, 1 \leq m_j \leq \lfloor \frac{n_j-1}{2} \rfloor$, the union of $t$ generalized Petersen graphs $P(n_j, m_j), j = 1, 2, \ldots, t$, has a vertex-magic total labeling with the magic constant $k = 10 \sum_{l=1}^{t} n_l + 2$.

**Proof.**

For all $i = 1, 2, \ldots, n_j$ and $j = 1, 2, \ldots, t$, label the vertices and edges of $\bigcup_{j=1}^{t} P(n_j, m_j)$ as follows.

$$
\lambda(u_i^j) = (n_j + 1 - i)\alpha(1, i - 1) + 1 + \sum_{l=1}^{j-1} n_l,
$$

$$
\lambda(v_i^j) = (m_j + 1 - i)\alpha(i, m_j) + 4\sum_{l=1}^{t} n_l + \sum_{l=1}^{j-1} n_l + (n_j + m_j + 1 - i)\alpha(m_j + 1, i),
$$

$$
\lambda(u_i^j u_{i+1}^j) = i + 3\sum_{l=1}^{t} n_l + \sum_{l=j+1}^{t} n_l,
$$

$$
\lambda(u_i^j v_i^j) = (n_j + 1 - i) + 2\sum_{l=1}^{t} n_l + \sum_{l=1}^{j-1} n_l,
$$

$$
\lambda(v_i^j v_{i+m_j}^j) = i + \sum_{l=1}^{t} n_l + \sum_{l=j+1}^{t} n_l,
$$
where
\[ \alpha(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x > y. \end{cases} \]

It is easy to verify that the labeling \( \lambda \) is a bijection from the set \( V(\bigcup_{j=1}^{t} P(n_j, m_j)) \) \( \cup \) \( E(\bigcup_{j=1}^{t} P(n_j, m_j)) \) onto the set \( 1, 2, \ldots, 5\sum_{l=1}^{t} n_l \).

Let us denote the weight (under labeling \( \lambda \)) of vertex \( u_j^i \) of \( \bigcup_{j=1}^{t} P(n_j, m_j) \) by
\[ w_{\lambda}(u_j^i) = \lambda(u_j^i) + \lambda(u_j^{i-1} u_j^i) + \lambda(u_j^i u_j^{i+1}) + \lambda(u_j^i v_j^i) \]
and the weights of vertex \( v_j^i \) by
\[ w_{\lambda}(v_j^i) = \lambda(v_j^i) + \lambda(v_j^{i-m_j} v_j^i) + \lambda(v_j^i u_j^{i+m_j}) + \lambda(u_j^i v_j^i). \]

Then, for \( i = 1, 2, \ldots, n_j \) and \( j = 1, 2, \ldots, t \), the labeling \( \lambda \) gives a vertex magic total labeling for \( \bigcup_{j=1}^{t} P(n_j, m_j) \) with the magic constant \( k = 10\sum_{l=1}^{t} n_l + 2 \). \( \square \)

Figure 1 and 2 show examples of vertex-magic total labeling on the union of 3 non-isomorphic and 4 non-isomorphic generalized Petersen graphs with the magic constant \( k = 234 \) and \( k = 282 \) respectively.

![Figure 1. VMTL of P(8, 3) ∪ P(8, 1) ∪ P(8, 2) with k = 234](image)

From duality, we can show that the construction of vertex-magic total labeling for union of \( t \) generalized Petersen graph is not unique. However, both labels have the same magic number \( k = 10\sum_{l=1}^{t} n_l + 2 \).
Fig. 2. VMTL of $P(7, 2) \cup P(8, 3) \cup P(7, 2) \cup P(6, 2)$ with $k = 282$

The union of $t$ special circulant graphs $\bigcup_{j=1}^{t} C_n(1, m_j)$ has a vertex set $V(\bigcup_{j=1}^{t} C_n(1, m_j)) = \{v^j_i | 0 \leq i \leq n-1, 1 \leq j \leq t\}$ and the edge set $E(\bigcup_{j=1}^{t} C_n(1, m_j)) = \{v^j_i v^j_{i+1} | 0 \leq i \leq n-1, 1 \leq j \leq t\} \cup \{v^j_i v^j_{i+m_j} | 0 \leq i \leq n-1, 1 \leq j \leq t\}$.

Special circulant graph $C_n(1, m)$ is a regular graph with $r = 4$. Theorem 1 states that if $G$ is a regular graph with $r$ is even, then $tG$ is vertex-magic whenever $t$ is an odd positive integer. Moreover, we construct that the union of $t$ special circulant graphs, $\bigcup_{j=1}^{t} C_n(1, m_j)$, for not only odd $t$, but also for even $t$. For the case of even $t$, our result is an example for $G$ regular and even $r$ that $tG$ can have vertex magic labeling.

**Theorem 3** For odd $n \geq 5$ and $m_j \in \{2, 3, \ldots, (n-1)/2\}$, the disjoint union of $t$ circulant graphs $\bigcup_{j=1}^{t} C_n(1, m_j)$ has a vertex magic total labeling with $k = 8tn + \frac{n-1}{2} + 3$.

**Proof.**

Let $\{C_n(1, m_j) : j = 1, \ldots, t\}$ be a set of circulant graphs with $n$ vertices. Let $\{v^j_i | i = 0, \ldots, n-1\}$ and $\{v^j_i v^j_{i+1} | i = 0, \ldots, n-1\} \cup \{v^j_i v^j_{i+m_j} | i = 0, \ldots, n-1\}$ be the sets of vertices and edges of $j^{th}$ circulant graphs $C_n(1, m_j)$, $j = 1, \ldots, t$, where $i + 1$ and $i + m_j$ are taken modulo $(n-1)$. Label the vertices and edges as follows

$$
\lambda(v^j_i) = n(j - \alpha(i, m_j - 1)) \text{ odd}(j) + n(2t - j + \alpha(m_j, i)) \text{ even}(j) + m_j - i,
$$

$$
\lambda(v^j_i v^j_{i+1}) = \frac{1}{2}(4tn + n \ j \text{ odd}(j) + n(2t - j + 1) \text{ even}(j) + n \text{ even}(i) - i),
$$

$$
\lambda(v^j_i v^j_{i+m_j}) = n(2t - j) \text{ odd}(j) + n(j - 1) \text{ even}(j) + i + 1,
$$

where

$$
\alpha(x, y) = \begin{cases} 
1 & \text{if } x \leq y \\
0 & \text{if } x > y,
\end{cases}
$$
odd \( (x) = \begin{cases} 1 & \text{if odd } x \\ 0 & \text{if another} \end{cases} \)

even \( (x) = \begin{cases} 1 & \text{if even } x \\ 0 & \text{if another} \end{cases} \)

Let us denote the weight of vertex \( v^j_i \) of \( \bigcup_{j=1}^{t} C_n(1, m_j) \) by

\[
 w_\lambda(v^j_i) = \lambda(v^j_i) + \lambda(v^j_{i-1}v^j_i) + \lambda(v^j_{i-1}v^j_{i+1}) + \lambda(v^j_{i-1}v^j_{i+1}v^j_{i+1}) + \lambda(v^j_i + m_j) + \lambda(v^j_{i+1}v^j_{i+1}v^j_{i+1})
\]

Then, for \( i = 0, 2, \ldots, n_j - 1 \) and \( j = 1, 2, \ldots, t \), the labeling \( \lambda \) gives a vertex magic total labeling for \( \bigcup_{j=1}^{t} C_n(1, m_j) \) with the magic constant \( k = 8tn + \frac{n-1}{2} + 3 \). □

Figure 3 shows an example of vertex-magic total labeling on the union of 4 non-isomorphic circulant graphs with the magic constant \( k = 295 \).

Fig. 3. VMTL of \( C_9(1, 2) \cup C_9(1, 3) \cup C_9(1, 4) \cup C_9(1, 3) \) with \( k = 295 \)

From duality property we can obtain Corollary 1.

**Corollary 1** For odd \( n \geq 5 \) and \( m_j \in \{2, 3, \ldots, (n-1)/2\} \), the disjoint union of circulants \( \bigcup_{j=1}^{t} C_n(1, m_j) \) has a vertex magic total labeling with \( k = 7tn - \frac{n-1}{2} + 2 \).

### 3 Conclusion

We conclude this paper with an open problem for further research direction in this area.

**Open Problem 1** Find if there is a vertex magic labeling for disjoint union of \( t \) (non-)isomorphic regular graphs other than \( \bigcup_{j=1}^{t} P(n_j, m_j) \) and \( \bigcup_{j=1}^{t} C_n(1, m_j) \).
References